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ON A QUASI-LINEAR EQUATION

Richard Bellman

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SUMMARY

The purpose of this paper is to establish some limit theorems for the solutions of $\cancel{x_i(n+1) = \max_q \sum_{j=1}^{n} a_{ij}(q)x_j(n).}$ non-linear recurrence relations. Recurrence relations of this kind occur in various dynamic programming problems. ( )

ON A QUASI-LINEAR EQUATION

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§1. Introduction

The purpose of this note is to establish some limit theorems for ~~the~~ non-linear recurrence relations

$$(1) \quad x_i(n+1) = \operatorname*{Max}_q \sum_{j=1}^{n} a_{ij}(q)x_j(n), \ j = 1, 2, \ldots, n, \ n \geq 0,$$

under certain assumptions concerning the initial values $c_i = x_i(0)$, and the coefficient matrices $A(q) = (a_{ij}(q))$.

Equations of this type occur in various parts of the theory of dynamic programming, as we shall indicate below, and are, in addition, of interest in furnishing a link between the theory of linear and non-linear operations, as we have discussed elsewhere, cf [1], [2], [3], [4].

§2. The Homogeneous Equation

Let us consider the equation

$$(1) \quad \lambda y_i = \operatorname*{Max}_q \sum_{j=1}^{n} a_{ij}(q)y_j, \ i = 1, 2, \ldots, n,$$

where we impose the following conditions.

(2) (a) $q = (q_1, q_2, \ldots, q_n)$ runs over some set of values, S, with the property that the maximum is attained in (1),

(b) $\infty > m \geq a_{ij}(q) > 0$, $i, j = 1,2,\ldots,n$ for $q \in S$,

(c) for any q, let $\phi(q)$ denote the characteristic root of $A(q) = (a_{ij}(q))$ of largest absolute value, the Perron

root, known to be positive. We assume that there exists at least one value of q for which $\phi(q)$ assumes its maximum for $q \in S$.

We shall now prove

Theorem 1. Under these conditions, there exists a unique positive λ with the property that (1) has a positive solution, $y_i > 0$, $i=1,2,\ldots,n$. This solution is unique up to a multiplicative constant, and

$$(3) \quad \lambda = \operatorname*{Max}_{q \in S} \phi(q).$$

Proof. We begin by showing the existence of a positive λ and a positive set of solutions $\{y_i\}$. Consider the region defined by $y_i \geq 0$, $\sum_{i=1}^{n} y_i = 1$. The normalized transformation

$$(4) \quad y_i' = \left[\operatorname*{Max}_{q} \sum_{j=1}^{n} a_{ij}(q) y_j \right] / \left[\sum_{i=1}^{n} \operatorname*{Max}_{q} \sum_{j=1}^{n} a_{ij}(q) y_j \right],$$

is a continuous mapping of this region into itself. Hence there exists a fixed point, $\{y_i\}$. This fixed point is a solution of (1), with λ the denominator in (4). Each component y_i is positive because of the positivity of $a_{ij}(q)$.

To show that this solution is unique up to a multiplicative constant, let $[\mu, z]$ be another solution of (1) with $\mu > 0$ and z a positive vector. Let $\{q\}$ be the set of values for which the maximum is attained in (1), and $\{\bar{q}\}$ the similar set associated with z. Observe that we may have different sets for each i. We have then

$$(5)\quad \lambda y_i = \sum_j a_{ij}(q)y_j \geq \sum a_{ij}(\bar{q})y_j, \quad i = 1, 2, \ldots, n,$$

$$\mu z_i = \sum_j a_{ij}(\bar{q})z_j.$$

Let us now assume, without loss of generality that $\lambda<\mu$. Let ϵ be a positive constant chosen so that one, at least, of the components $y_i-\epsilon z_i$ is zero, one at least is positive, and the others are non-negative. This can always be accomplished if y and z are not proportional. If i is an index for which $y_i-\epsilon z_i$ is zero, we have

$$(6)\quad 0 = \mu(y_i-\epsilon z_i) > \lambda y_i-\epsilon\mu z_i \geq \sum_{j=1}^{n} a_{ij}(\bar{q})(y_j-\epsilon z_j) > 0,$$

Since $a_{ij}(\bar{q})>0$, a contradiction. Hence $\lambda=\mu$ and y and z are proportional.

To show that $\lambda = \underset{q}{\operatorname{Max}}\ \phi(q)$, we proceed as follows. Let $\mu = \underset{q}{\operatorname{Max}}\ \phi(q)$. It is clear that λ, as the characteristic root of some $A(q)$, satisfies the inequality $\lambda\leq\mu$. Assume that actually $\mu>\lambda$. Let $z = (z_1,z_2,\ldots,z_n)$ be a positive characteristic vector associated with μ and $\bar{q}$ a set of q-values which yield $\mu = \phi(\bar{q})$. Then we have

$$(7)\quad \mu z_i = \sum_{j=1}^{n} a_{ij}(\bar{q})z_j \leq \underset{q}{\operatorname{Max}} \sum_{j=1}^{n} a_{ij}(q)z_j$$

Since y_i is positive, we can find a positive constant m such that $z_i\leq my_i$ for $i = 1,2,\ldots,n$. Hence, (7) yields

$$(8)\quad \mu z_i \leq m\ \underset{q}{\operatorname{Max}} \sum_{j=1}^{n} a_{ij}(q)y_j = m\lambda y_i$$

Thus $z_i \leq my_i \lambda/\mu$. Iterating this we obtain $z_i \leq my_i(\lambda/\mu)^k$, for arbitrary k. Since $\lambda/\mu < 1$, by assumption, this yields $z_i = 0$, a contradiction. Hence $\lambda = \mu$.

§3. The Recurrence Relation

Let us now return to the recurrence relation of (1.1) and prove

Theorem 2. If, in addition to the conditions of (2.2), we assume that there is a unique q for which the maximum value of $\phi(q)$ is attained and that $c_i \geq 0$, then

(1) $x_i(n) \sim ay_i \lambda^n$,

as $n \to \infty$, where a is a constant dependent upon the initial values c_i.

Proof. Let us take $c_i > 0$ without loss of generality. There are then two positive constants k and K such that $ky_i \leq c_i \leq Ky_i$, $i=1,2,\ldots,n$. Let us show inductively that

(2) $ky_i \lambda^n \leq x_i(n) \leq Ky_i \lambda^n$.

Assume that we have the result for n, then

(3)
$$\begin{aligned} x_i(n+1) &\leq K\lambda^n \operatorname*{Max}_q \sum_{j=1}^{n} a_{ij}(q)y_j = K\lambda^{n+1}y_i \\ &\geq k\lambda^n \operatorname*{Max}_q \sum_{j=1}^{n} a_{ij}(q)y_j = k\lambda^{n+1}y_i. \end{aligned}$$

To establish the asymptotic behavior we show that for n sufficiently large the set of q's which furnish the maximum in (1.1) is precisely the set which yields $\lambda = \operatorname{Max} \phi(q)$.

Assume the contrary. This means that infinitely often we employ a set $\{\bar{q}\}$ which is not identical with the q which furnishes the maximum in $\phi(q)$.

We then have

$$(4)\quad x_i(n+1) = \sum_{j=1}^{n} a_{ij}(\bar{q})x_j(n),\ i=1,2,\ldots,n$$
$$\leq \left(\sum_{j=1}^{n} a_{ij}(\bar{q})y_j\right) K\lambda^n.$$

For some index i we must have

$$(5)\quad \sum_{j=1}^{n} a_{ij}(\bar{q})y_j < \lambda y_i,$$

with strict inequality. For if $\sum_{j=1}^{n} a_{ij}(\bar{q})y_j \geq \lambda y_i$ for all i, the characteristic root of $A(\bar{q}) = (a_{ij}(\bar{q}))$ of largest absolute value, $\phi(\bar{q})$, would at least equal $\lambda = \underset{q}{\text{Max}}\ \phi(q)$, which would contradict the assumption concerning the uniqueness of the maximum of $\phi(q)$.

Hence, for some component, say the first, we have

$$(6)\quad x_1(n+1) \leq \theta K\lambda^{n+1}y_1,\ 0 < \theta < 1.$$

Since $a_{ij}(\overset{*}{q}) > 0$ for i,j, where $\overset{*}{q}$ is the value of q for which $\lambda = \phi(\overset{*}{q})$, we see that, for i = 1,2,...,n,

$$(7)\quad x_i(n+2) \leq K\lambda^{n+1} \left[\sum_{j=2}^{n} a_{ij}(\overset{*}{q})y_i + \theta a_{1j}(\overset{*}{q})y_1\right]$$
$$\leq \theta_1 K\lambda^{n+2}y_i,$$

where $\theta < 1$.

If therefore a set of q's distinct from $\overset{*}{q}$ are used R times, we obtain

(8) $x_1(n) \leq \theta_1{}^R \, K\lambda^n y_1$,

for n sufficiently large. Since $0 < \theta_1 < 1$, if R is too large we eventually contradict the lower bound for $x_1(n)$.

Hence for $n \geq n_0 = n_0(c_1)$, we have

(9) $x(n+1) = A(\overset{*}{q})\, x(n)$,

whence the asymptotic statement of (1) follows.

§4. <u>A Dynamic Programming Problem</u>

Suppose that we are engaged in a multi-stage decision process of the following type. At each stage we have our choice of various operations, which we number $i=1,2,\ldots,K$. The $i^{\underline{th}}$ operation has a probability distribution attached with the following properties:

(1) a. There is a probability p_{ik} that we receive k units and the process continues, $k=1,2,\ldots,R$,

b. There is a probability p_{i0} that we receive nothing and the process terminates.

How do we proceed so as to maximize the probability that we receive at least n units before the process terminates?

Let us define the sequence

(2) $u(n)$ = the probability of attaining at least n units before the termination of the process using an optimal procedure.

Then using the intuitive "principle of optimality", cf. [1], [2], [3], [4], we see that $u(n)$ satisfies the recurrence relation

$$(3)\quad u(n) = \underset{i}{\mathrm{Max}} \left[\sum_{k=1}^{R} p_{ik} u(n-k) \right], \quad n > 0$$
$$= 1, \qquad n \leq 0.$$

Using methods similar to those above, we see that for large n,

$$(4)\quad u(n) \sim c\rho^n,$$

where ρ is the root of largest absolute value, necessarily positive, of

$$(5)\quad 1 = \sum_{k=1}^{R} p_{ik} \rho^{-k},$$

for the value of i which maximizes ρ.

§5. An Analogue of a Result of Markoff

Markoff showed that if

$$(1)\quad x_i(n+1) = \sum_{j=1}^{n} a_{ij} x_j(n), \quad n = 0,1,\ldots,$$

$x_i(0)>0$, with the conditions

$$(2)\quad a_{ij}>0, \ \sum_j a_{ij} = 1, \ i = 1,2,\ldots,n,$$

then

$$(3)\quad \lim_{n \to \infty} x_i(n) = c, \ i = 1,2,\ldots,n,$$

where c depends on the initial values.

The same proof shows that the same result holds for the sequence defined by

$$(4) \quad x_i(n+1) = \operatorname*{Max}_q \sum_{j=1}^{n} a_{ij}(q) x_j(n),$$

provided that the conditions in (2) hold uniformly in q. The constant will, of course, in general, be different from that above.

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